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CHARACTERISTICS FOR GENERALIZED HERMOMECHANICAL FIELDS IN VISCOUS FLUID MEDIA

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In the context of a general theory of characteristics, thermomechanical processes are considered in fluid media with the final heat-propagation velocity. The equations of propagation of characteristics for two variants of the theory of a viscous fluid medium are derived, and ways of applying them for determining displacement velocities are indicated.

The theory of nonstationary viscous-fluid flows has been studied rather adequately in the classical case assuming the symmetry of stress tensors [1]. Therefore, of interest is investigating the regularities of wave surfaces in a more general asymmetric case with account for the final heat-propagation velocity. In the present paper, we consider one particular case of this great problem associated with a study of the regularities of the existence of the characteristic surfaces for disconnected thermomechanical fields. Interest in this problem is motivated by the fact that the characteristics represent the surfaces of the propagation of discontinuities of the velocity and temperature fields.

In the case where the dissipation of energy can be neglected, the thermomechanical fields in viscous fluid media are described by a nonlinear system of hyperbolic equations composed of the heat-conduction equation and equations of motion, continuity, and state [1-3]:

$$\begin{cases} \alpha_1 \Delta\theta = \dfrac{d\theta}{dt} + \tau_{r1} \dfrac{d^2\theta}{dt}\,, \\ \mu\Delta\vec{v} + (\lambda + \mu)\,\mathrm{grad\,div}\,\vec{v} - \mathrm{grad}\,p = \rho\,\dfrac{d\vec{v}}{dt}\,, \\ \dfrac{d\rho}{dt} + \mathrm{div}\,\rho\vec{v} = 0\,, \\ p = f(\rho, T)\,. \end{cases}$$

In view of the complexity of this system, we will consider a simplified variant of it, assuming that the fluid is incompressible (ρ = const) and motions occur at small velocities:

$$\alpha_1 \Delta\theta - \frac{\partial\theta}{\partial t} - \tau_{r1} \frac{\partial^2\theta}{\partial t} = 0\,, \quad \mu\Delta\vec{v} - \mathrm{grad}\,p = \rho\,\frac{\partial\vec{v}}{\partial t}\,, \quad \mathrm{div}\,\vec{v} = 0\,. \tag{1}$$

Here λ and μ are, respectively, the volume and shear viscosities, $\vec{v} = (v_1, v_2, v_3)$ are the components of the velocity vector, p is the pressure, α_1 is the thermal conductivity coefficient, τ_{r1} is the relaxation time of the heat flux, and θ is the temperature increment.

We add the following initial conditions to Eq. (1):

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$$v_i|_{t=0}=f_i(0,x)\,,\quad p|_{t=0}=\varphi(0,x)\,,\quad \theta|_{t=0}=\xi(0,x)\,,\quad \left.\frac{\partial\theta}{\partial t}\right|_{t=0}=\psi(0,x)\,. \tag{2}$$

If the initial conditions are analytic functions and the plane $t = 0$ is not characteristic, problem (1)-(2) has a unique solution. Therefore, the incorrectness of a Cauchy problem, in particular, its unsolvability [4-6], will be a criterion for finding the characteristic surfaces.

We use this fact as applied to system (1); for this, we go over in Eq. (1) to new variables following the scheme from [4]: $\begin{pmatrix} t=x_0, & x_1, x_2, x_3 \\ Z, & Z_1, Z_2, Z_3 \end{pmatrix}$. In this case

$$\frac{\partial(*)}{\partial x_k}=\frac{\partial(*)}{\partial Z}\frac{\partial Z}{\partial x_k}+\sum_{i=1}^{3}\frac{\partial(*)}{\partial Z_i}\frac{\partial Z_i}{\partial x_k}\,;$$

$$\frac{\partial^2(*)}{\partial x_k\partial x_l}=\frac{\partial^2(*)}{\partial Z^2}\frac{\partial Z}{\partial x_k}\frac{\partial Z}{\partial x_l}+\frac{\partial(*)}{\partial Z}\frac{\partial^2 Z}{\partial x_k\partial x_l}+\sum_{i,j=1}^{3}\frac{\partial^2(*)}{\partial Z_i\partial Z_j}\frac{\partial Z_i}{\partial x_k}\frac{\partial Z_j}{\partial x_l}+\sum_{i=1}^{3}\frac{\partial(*)}{\partial Z_i}\frac{\partial^2 Z_i}{\partial x_k\partial x_l}\,. \tag{3}$$

By virtue of Eq. (3), Eq. (1) takes the form (we write the higher-order terms in the system transformed):

$$\frac{\partial^2\theta}{\partial Z^2}\sum_{k=1}^{3}\left(\frac{\partial Z}{\partial x_k}\right)^2-\frac{\tau_{r1}}{\alpha_t}\frac{\partial^2\theta}{\partial Z^2}\left(\frac{\partial Z}{\partial t}\right)^2+\ldots=0\,,$$

$$\mu\frac{\partial v_i}{\partial Z}\sum_{k=1}^{3}\frac{\partial^2 Z}{\partial x_k^2}-\frac{\partial p}{\partial Z}\frac{\partial Z}{\partial x_i}+\ldots=\rho\frac{\partial v_i}{\partial Z}\frac{\partial Z}{\partial t}\,,\quad \sum_{k=1}^{3}\frac{\partial v_k}{\partial Z}\frac{\partial Z}{\partial x_k}+\ldots=0\,. \tag{4}$$

The unsolvability condition of the Cauchy problem for system (4) is equivalent to the zero equality of the determinant [5, 6] that is composed of the coefficients at the derivatives $\frac{\partial v_i}{\partial Z}$, $\frac{\partial^2\theta}{\partial Z^2}$, and $\frac{\partial p}{\partial Z}$. This leads to the following equation of the characteristics:

$$\begin{vmatrix} g^2-\frac{\tau_{r1}}{\alpha_t}\left(\frac{\partial Z}{\partial t}\right)^2 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & -\frac{\partial Z}{\partial x_1} \\ 0 & 0 & A & 0 & -\frac{\partial Z}{\partial x_2} \\ 0 & 0 & 0 & A & -\frac{\partial Z}{\partial x_3} \\ 0 & -\frac{\partial Z}{\partial x_1} & -\frac{\partial Z}{\partial x_2} & -\frac{\partial Z}{\partial x_3} & 0 \end{vmatrix}=0\,. \tag{5}$$

Here we have introduced the notation

$$g^2 = \sum_{k=1}^{3} \left(\frac{\partial Z}{\partial x_k}\right)^2 ; \quad A = \mu \Delta Z - \rho \frac{\partial Z}{\partial t} .$$

Expanding this determinant, we have

$$g^2 A^2 \left(g^2 - \frac{\tau_{r1}}{\alpha_t} \left(\frac{\partial Z}{\partial t}\right)^2 \right) = 0 . \tag{6}$$

Hence the following equations for determining the propagation velocities of the characteristics are obtained:

$$A = 0 , \tag{7}$$

$$g^2 - \frac{\tau_{r1}}{\alpha_t} \left(\frac{\partial Z}{\partial t}\right)^2 = 0 . \tag{8}$$

Taking into account that the displacement velocity of the surface of discontinuity [5, 6] is $P = \frac{1}{g}\frac{dZ}{dt}$, from Eq. (8) we have $P = \sqrt{\alpha_t / \tau_{r1}}$. Moreover, from Eq. (6) it follows that $g^2 = 0$ is one of the solutions that corresponds to the stationary characteristic surface (the existence of these surfaces is proved experimentally [7]).

Now we consider the generalization of this theory to the case of a semimoment viscous fluid medium in which the rotation of a local trihedron is equal to the mean rotation of the displacement field:

$$\vec{\omega} = \frac{1}{2} \operatorname{rot} \vec{v}, \ \vec{\omega} = (\omega_1, \omega_2, \omega_3) , \quad \vec{v} = (v_1, v_2, v_3) .$$

The equations of motion can be written in this form [8-10]:

$$\begin{gathered} \sum_{j=1}^{3} \sigma_{ji,j} + X_i = \rho \frac{dv_i}{dt} , \\ \sum_{j=1}^{3} \mu_{ji,j} + \sum_{j,k=1}^{3} \varepsilon_{ijk} \sigma_{jk} + Y_i = j \frac{d\omega_i}{dt} . \end{gathered} \tag{9}$$

The tensors of force and moment stresses have symmetric and nonsymmetric parts and are determined by the formulas [8-10]

$$\sigma_{ij} = \mu (\partial_i v_j + \partial_j v_i) + \alpha (\partial_j v_i - \partial_i v_j) - 2\alpha \sum_{k=1}^{3} \varepsilon_{ijk} \omega_k + \left(\lambda \sum_{k=1}^{3} \partial_k v_k - p \right) \delta_{ij} ,$$

$$\mu_{ij} = \gamma (\partial_j \omega_i + \partial_i \omega_j) + \beta (\partial_j \omega_i - \partial_i \omega_j) .$$

The nonsymmetric part of the tensor of force stresses $\sigma_{(ij)} = \frac{1}{2}(\sigma_{ij} + \sigma_{ji})$ can be found from the second equation of system (9):

$$\sigma_{\langle lm \rangle} \equiv \frac{1}{2} (\sigma_{lm} - \sigma_{ml}) = -\frac{1}{2} \sum_{i,j=1,3} \varepsilon_{ilm} \mu_{ji,j} - \frac{1}{2} \sum_{i=1}^{3} \varepsilon_{ilm} \left(Y_i - j \frac{d\omega_i}{dt} \right) .$$

Then

$$\sigma_{(ij)} = \frac{1}{2}(\sigma_{ij} + \sigma_{ji}) = \alpha\,(\partial_j v_i - \partial_i v_j) - 2\alpha \sum_{k=1}^{3} \varepsilon_{ijk}\omega_k \,.$$

The substitution of the components of the tensors of force and moment stresses into the first equation of system (3) gives

$$\mu\Delta\vec{v} + (\lambda + \mu)\,\mathrm{grad\,div}\,\vec{v} - \mathrm{grad}\,p + \frac{1}{4}(\gamma + \beta)\,\mathrm{rot\,rot}\,\Delta\vec{v} +$$

$$+ \frac{1}{2}\,\mathrm{rot}\left(\vec{Y} - j\frac{d\vec{\omega}}{dt}\right) + \vec{X} = \rho\frac{d\vec{v}}{dt} \,.$$

With allowance for the fact that $\vec{\omega} = \frac{1}{2}\,\mathrm{rot}\,\vec{v}$, we have

$$\mu\Delta\vec{v} + (\lambda + \mu)\,\mathrm{grad\,div}\,\vec{v} - \mathrm{grad}\,p + \frac{1}{4}(\gamma + \beta)\,\mathrm{rot\,rot}\,\Delta\vec{v} +$$

$$+ \frac{1}{2}\,\mathrm{rot}\,\vec{Y} + \vec{X} = \rho\frac{d\vec{v}}{dt} + \frac{j}{4}\,\mathrm{rot}\left[\frac{d}{dt}(\mathrm{rot}\,\vec{v})\right] . \tag{10}$$

To this equation we add that of continuity:

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}\,\vec{v} = 0 \,. \tag{11}$$

In order to describe the dynamic temperature stresses in viscous fluids, we use a generalized hyperbolic heat-conduction equation for an isotropic viscous medium [2]:

$$\beta_t\Delta\theta = \frac{d\theta}{dt} - \tau_{r2}\frac{d^2\theta}{dt^2} \,. \tag{12}$$

In this form, the system of equations (10)-(12) is a resolving system of equations for the variant considered of the asymmetric theory of a viscous thermomechanical medium. For its simplicity we assume that the motions occur at small velocities and the fluid is incompressible (ρ = const). Then we will have

$$\begin{aligned} &\mu\Delta\vec{v} - \frac{1}{4}(\gamma + \beta)\,\Delta^2\vec{v} + \frac{\partial}{\partial t}\left(\frac{j}{4}\Delta - \rho\right)\vec{v} - \mathrm{grad}\,p = -\frac{1}{2}\,\mathrm{rot}\,\vec{Y} - \vec{X}, \\ &\mathrm{div}\,\vec{v} = 0 \,, \\ &\beta_t\Delta\theta - \frac{\partial\theta}{\partial t} - \tau_{r2}\frac{\partial^2\theta}{\partial t^2} = 0 \,. \end{aligned} \tag{13}$$

The initial data for this problem have just the same form as for system (1).

The equation of the characteristic surface can be found from the condition that from system (13), written in the new variables Z, Z_1, Z_2, and Z_3, it is impossible to determine $\frac{\partial v_i}{\partial Z}$, $\frac{\partial p}{\partial Z}$, and $\frac{\partial^2\theta}{\partial Z^2}$.

It should be noted that if $v_i\,|_{t=0} = f_i(0, X)$, then we also have $\Delta v_i\,|_{t=0} = \Delta f_i(0, X)$, and this means that $(\vec{v} - \frac{j}{4}\Delta\vec{v})$ is the sought function for which the Cauchy problem can be considered.

From the aforesaid the equation for the characteristic surfaces follows in such a form:

$$\begin{vmatrix} B & 0 & 0 & -\dfrac{\partial Z}{\partial x_1} & 0 \\ 0 & B & 0 & -\dfrac{\partial Z}{\partial x_2} & 0 \\ 0 & 0 & B & -\dfrac{\partial Z}{\partial x_3} & 0 \\ 0 & 0 & 0 & 0 & g^2 - \dfrac{\tau_{r2}}{\beta_t}\left(\dfrac{\partial Z}{\partial t}\right)^2 \\ -\dfrac{\partial Z}{\partial x_1} & -\dfrac{\partial Z}{\partial x_2} & -\dfrac{\partial Z}{\partial x_3} & 0 & 0 \end{vmatrix} = 0,$$

where

$$B = \mu\Delta Z - \frac{1}{4}(\gamma + \beta)\,\Delta^2 Z + \frac{j}{4}\frac{\partial}{\partial t}\Delta Z - \rho\frac{\partial}{\partial t} Z .$$

Resolving the determinant, we will have

$$g^2 B^2 \left(g^2 - \frac{\tau_{r2}}{\beta_t}\left(\frac{\partial Z}{\partial t}\right)^2\right) = 0,$$

whence the following equations for the propagation of the characteristics are obtained:

$$\mu\Delta Z - \frac{1}{4}(\gamma + \beta)\,\Delta^2 Z + \frac{j}{4}\frac{\partial}{\partial t}\Delta Z - \rho\frac{\partial Z}{\partial t} = 0, \tag{14}$$

$$g^2 - \frac{\tau_{r2}}{\beta_t}\left(\frac{\partial Z}{\partial t}\right)^2 = 0, \tag{15}$$

$$g^2 = 0. \tag{16}$$

Equation (14) with $j = 0$ and $\gamma + \beta = 0$ gives the characteristic equation for the classical theory of viscous fluids. We should also note the fact that the characteristic equations for both the classical and semimoment theories of viscous deformable fluid media determine the wave surfaces of quasi-elastic and quasithermal nonstationary discontinuities [11]. Moreover, even in the general case where the fluid is compressible and the effect of connectedness between the $\vec{v}$ and θ fields is present no surface of discontinuity of a thermoelastic nature occurs. All the aforesaid can be extended to the more general case of the asymmetric mechanics of a viscous medium for both small and final velocities of wave motions.

Formulas (7) and (14) allow us to write dispersion relations for the classical and semimoment hydrodynamics. For this, we seek solutions of these equations of the plane-wave type $v = v_0 \exp(-i(\omega t - \vec{k}\vec{r}))$, which leads to the following equations:

$$\mu k^2 - i\rho\omega = 0,$$

$$\mu k^2 + (\gamma + \beta)\,k^4/4 = i\omega k^2 + i\omega\rho .$$

These equations have an explicit character, which makes it possible to solve them numerically on a personal computer.

NOTATION

α, λ, μ, β, and γ, material constants of the liquid medium; p, pressure; X_i and Y_i, internal forces and distributed moments; j, moment of inertia; ε_{ijk}, Levi-Civita symbol; δ_{ij}, Kronecker symbol; $\vec{v}$, linear velocity; $\vec{\omega}$, velocity of rotational motion; θ, increment in the temperature compared to the temperature of the natural state T_0; ρ, medium density; $\tau_{r1,r2}$, relaxation time of the heat flux; α_t and β_t, thermal conductivity coefficients; i, imaginary unit; $\vec{k}$, wave vector; ω, cyclic frequency. Subscripts: t, heat; r, relaxation.

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